Lecture 1 : Precalculus Review

Hyperlinks are shown in blue, download the `cdf player` *from the Wolfram Alpha website to view the* `Wolfram Alpha interactive demonstrations`*. When you have downloaded the cdf player, click on this symbol to view the demonstration.*

Functions

(A more detailed review can be found in `Lecture 10` *under* `Algebra/Precalculus Review / Lectures for Calc. 1 Prep.` *on our website.)*

A function arises when one quantity depends on another. Many everyday relationships between variables can be expressed as functions.

Example Consider the volume of a cylindrical glass with radius = 1 inch. We have a formula for the volume; $V = \pi r^2 h = \pi h \ \ in^3$, where h is the height of the glass in inches and $r = 1$ is the radius.

We see that the value of the volume depends on the height, h; V is a function of h. We sometimes indicate that the value of V depends on the value of h, by writing the formula as $V(h) = \pi h \ \ in^3$.

When $h = 2$, $V = V(2) =$ and when $h = 3, V = V(3) =$

Example The cost of (short-term) parking at South Bend airport depends on how long you leave your car in the short term lot. The parking rates are described in the following table:

First 30 minutes	Free
31-60 minutes	\$2
Each additional hour	\$2
24 hour maximum rate	\$13

The Cost of Parking is a function of the amount of time the car spends in the lot. If we are to create a formula for the cost of parking = C, in terms of how long our car stays in the lot = t, we need to give the formula piece by piece as follows:

$$C(t) = \begin{cases} \$0 & 0 \le t \le 0.5\ hr. \\ \$2 & 0.5\ hr. < t \le 1\ hr. \\ \$4 & 1\ hr. < t \le 2\ hr. \\ \$6 & 2\ hr. < t \le 3 hr. \\ \$8 & 3\ hr. < t \le 4\ hr. \\ \$10 & 4\ hr. < t \le 5\ hr. \\ \$12 & 5\ hr. < t \le 6\ hr. \\ \$13 & 6\ hr. < t \le 24\ hr. \\ \$13 + \text{cost of towing} & t > 24\ hr. \end{cases}$$

This is an example of a **piecewise defined function.**

The formal definition of a function is as follows:

Definition A function f is a <u>rule</u> which assigns to each element x of a set D, exactly one element, $f(x)$, of a set E. The **Domain** of a function f is the set D, the set of all values of x, for which $f(x)$ is defined. The **Range** of the function f is the set of all elements of the target set E which have the form $f(x)$, for some x in D.

We can represent functions in **four ways**, **verbally**(a description in words), **visually**(a set theoretic picture or a graph on the xy-plane), **numerically**(using a table of values) and **algebraically**(by an explicit formula).

Algebraic Representation of Functions

All of the functions that we will consider in this course will have a numerical domain and range. Most can be described by giving a a formula for $f(x)$, as in Examples 1 and 2. If we are given a formula for a function f, unless otherwise specified, we assume that the **domain** is the set of all real numbers for which the formula makes sense; domain of $f = \{x \in \mathbb{R} | f(x) \text{ exists}\}$. The **range** is the set of all numbers of the form $f(x)$, range of $f = \{f(x) | x \in \text{Domain of } f\}$.

Example Let $f(x) = \frac{x^2}{x-1}$. What is the domain of f?

What is $f(1 + \frac{1}{100})$?

What is $f(1 + h)$?

What is $\frac{f(0+h)-f(0)}{h}$?

Note that $f(x) = |x|$ is really a piecewise defined function with formula:

$$f(x) = \begin{cases} -x & x \leq 0 \\ x & x > 0 \end{cases}.$$

Example Write $h(x) = |x - 1|$ as a piecewise defined function.

Graphs on the Cartesian Plane

(A more detailed review can be found in your online homework under ***`graphing and functions`*** *and in* `Lecture 8` *and* `Lecture 11` *under* `Algebra/Precalculus Review / Lectures for Calc. 1 Prep.` *on our website.)*

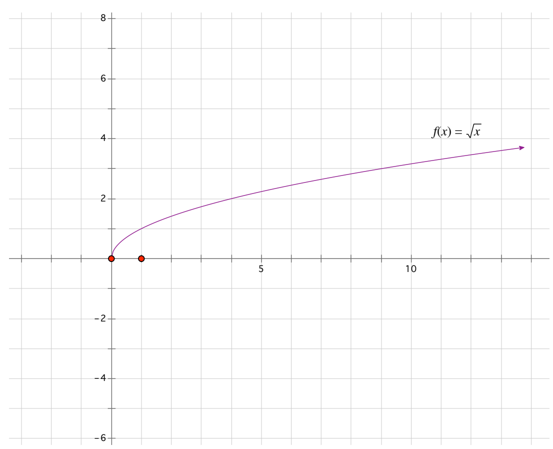
When we give a formula, $f(x)$, describing the value of f at any point x in the domain, the values of x vary over all values in the domain. Hence x **is a variable, called the independent variable**. With each function, f, described by such a formula $f(x)$, we can **associate an equation** $y = f(x)$. For example, if $f(x) = \sqrt{x}$, then the associated equation is $y = \sqrt{x}$. Thus we create a new variable y, whose values vary over the values in the range of the function. Note the value of y depends on the value of x, hence y **is called a dependent variable. The graph of a function f, with associated formula $f(x)$, is the set of all points (x, y) on the xy-plane that satisfy the equation $y = f(x)$.** This is just a picture of all points of the form $(x, f(x))$ on the plane.

You should already be very familiar with the graphs shown in the catalogue at the end of the lecture.

For any function, every value in the domain gives one and only one value in the range when we apply the formula, as a result, each vertical line in the xy-plane cuts the graph of a function at most once. In fact, as we will see below, this is the defining characteristic of a graph of a function.

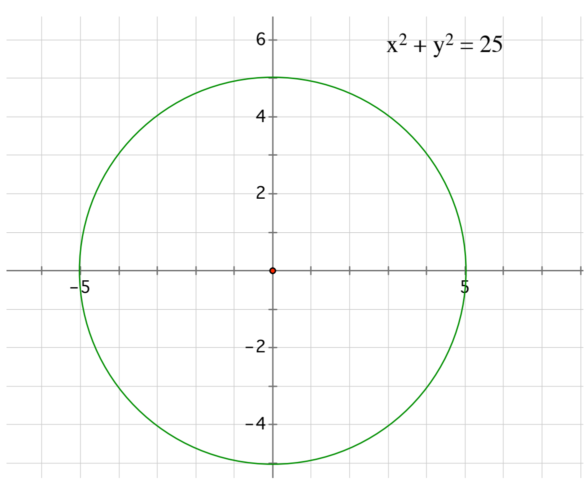
The domain of a function can be identified from the graph as the set of all values of x for which a vertical line through x cuts the graph. The range of a function can be identified from the graph as the set of all values of y for which a horizontal line through y cuts through the graph.

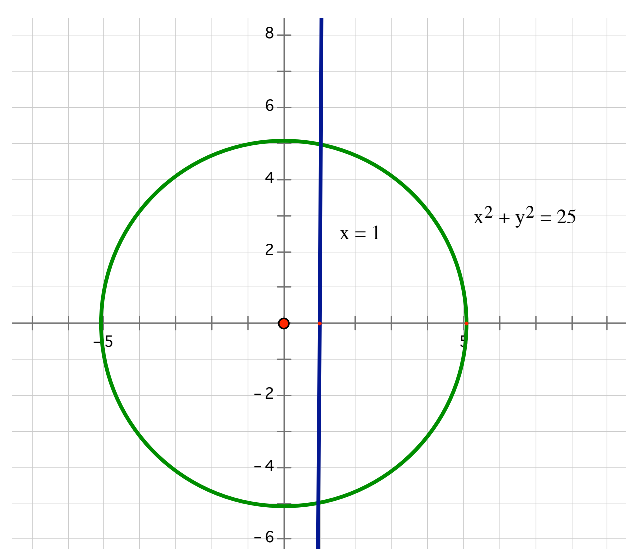
Example If $f(x) = \sqrt{x}$, use the graph of the function to identify the domain and range of f?



Graphing general equations : We can plot the points on any curve defined by an equation in x and y on the xy-plane. However, we cannot always solve for y uniquely in such equations and the graph may not be the graph of a function of the form $y = f(x)$.

Example the graph of the equation of a circle of radius 5 with center at $(0, 0)$ is shown on the left below:



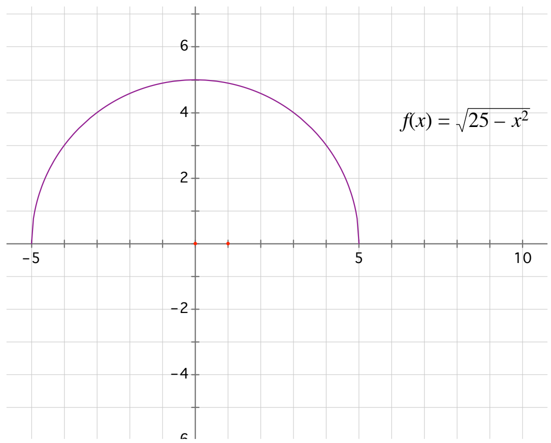


Vertical Line Test (VLT) The graph of an equation passes the vertical line test if each vertical line cuts the graph at most once.

If the graph of an equation passes the Vertical Line Test (VLT) then it is the graph of a function (to each x-value, there corresponds at most one y-value on the graph), if we can solve for y in terms of x, we will get the equation corresponding to the function: $y = f(x)$.

Clearly the graph of the equation $x^2 + y^2 = 25$ does not pass the vertical line test (see the vertical line drawn in the picture on the right above)

On the other hand the upper half of the above circle does pass the VLT and is the graph of the function, $f(x) = \sqrt{25 - x^2}$.



Graphing Techniques

(A more detailed review can be found in `Lecture 12` *under* `Algebra/Precalculus Review / Lectures for Calc. 1 Prep.` *on our website.)*

We can use our small catalogue of graphs from the appendix to graph many functions, by taking note of how graphs change with the transformations listed below:

Vertical and Horizontal Shifts

VERTICAL AND HORIZONTAL SHIFTS Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward

$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward

$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right

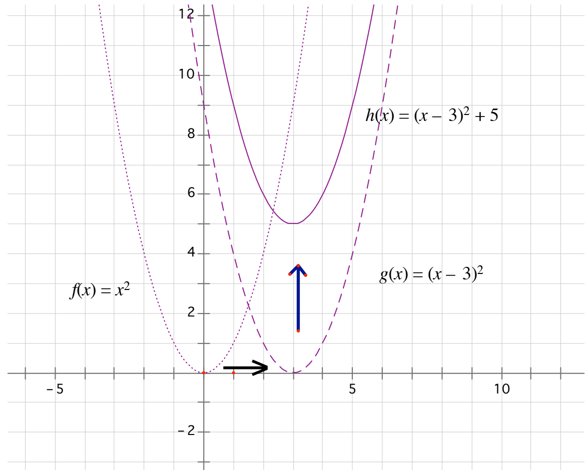
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Example Sketch the graph of the equation $y = x^2 - 6x + 14$.

We complete the square to transform this equation to

$$y = (x - 3)^2 + 5.$$

We see that the graph is the graph of $y = x^2$, shifted, first, 3 units to the right and then shifted 5 units upwards:



We also have a number of transformations which stretch, compress or reflect (in a line) the the graph of a function:

VERTICAL AND HORIZONTAL STRETCHING AND REFLECTING Suppose $c > 1$. To obtain the graph of

- $y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
- $y = (1/c)f(x)$, compress the graph of $y = f(x)$ vertically by a factor of c
- $y = f(cx)$, compress the graph of $y = f(x)$ horizontally by a factor of c
- $y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
- $y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis
- $y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

In the Appendix, it is demonstrated how to use these techniques to graph $y = 2\sin(4x)$.

Increasing/Decreasing Functions

Definition We say a function $f(x)$ is increasing on the interval $[a, b]$, if the graph is moving upwards from left to right. Algebraically, this amounts to the statement:
A function f is called **increasing** on the interval $[a, b]$ if $f(x_1) < f(x_2)$ for any two numbers x_1 and x_2 with $a \leq x_1 < x_2 \leq b$.
Similarly we say the function $f(x)$ is decreasing on the interval $[a, b]$ if the graph of f is moving downwards from left to right on that interval
or A function f is **decreasing** on the interval $[a, b]$ if $f(x_1) > f(x_2)$ for any two numbers x_1 and x_2 with $a \leq x_1 < x_2 \leq b$.

Example Use the graph of $y = x^2$ to find the intervals where f is increasing and decreasing.

Operations on Functions

(A more detailed review can be found in `Lecture 13` *under* `Algebra/Precalculus Review / Lectures for Calc. 1 Prep.` *on our website.)*

As well as being able to add, subtract, multiply and divide functions, we can form the composition of two functions f and g.

Definition Given two functions f and g, the composite function $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x such that

1. x is in the domain of g and
2. $g(x)$ is in the domain of f

It is implicit in the definition of the function, that we first calculate $g(x)$ and then apply f to the result. Hence any x in the domain must satisfy the above two conditions.
(When calculating the domain, calculating just the formula for $f \circ g$ or $g \circ f$ can be misleading)

Example If $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$,
then $(f \circ g)(x) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$.
On the other hand, $(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 + 1 = x + 1$.

The domain of f is $\{x \in \mathbb{R} | x \geq 0\}$ and the domain of g is the set of all real numbers.

The domain of $(f \circ g)(x)$ is the set of all x such that

1. x is in the domain of $g(x) = x^2 + 1 \quad -> \quad x$ is any real number AND
2. $g(x)$ is in the domain of $f \quad -> \quad g(x) = x^2 + 1 \geq 0 \quad -> \quad$ this is true for any real number x.

Hence the domain of $f \circ g$ is the set of all real numbers, $\mathbb{R}$.

The domain of $(g \circ f)(x)$ is the set of all x such that

1. x is in the domain of $f \quad -> \quad x \geq 0 \quad$ AND
2. $f(x)$ is in the domain of $g \quad -> \quad \sqrt{x}$ is a real number $\quad -> \quad$ this is true for all $x \geq 0$.

Hence the domain of $g \circ f(x)$ is the set of all nonnegative real numbers :$\{x \in \mathbb{R} | x \geq 0\}$ despite the fact that all real numbers make sense in the final formula. When describing such a function with a restricted domain, we use a piecewise description

$$(g \circ f)(x) = \begin{cases} x + 1 & x \geq 0 \\ \text{undefined} & \text{otherwise} \end{cases}.$$

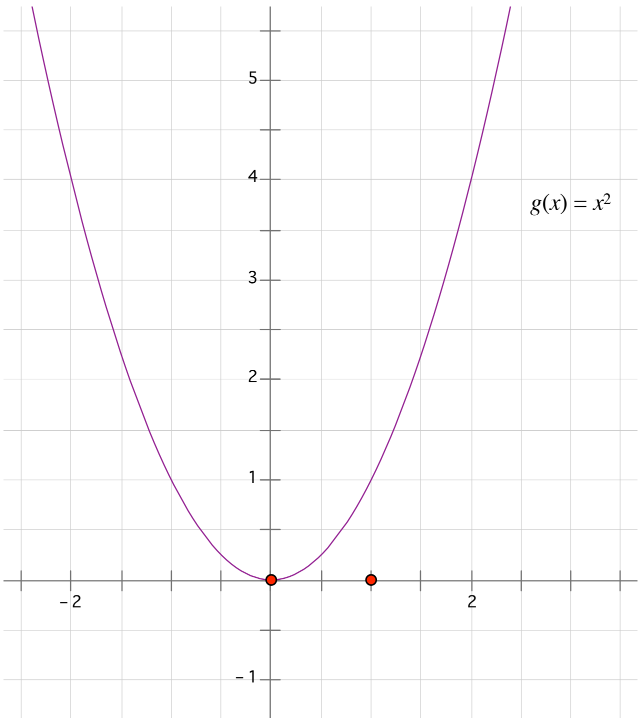
Partial information about Functions and Models of functions

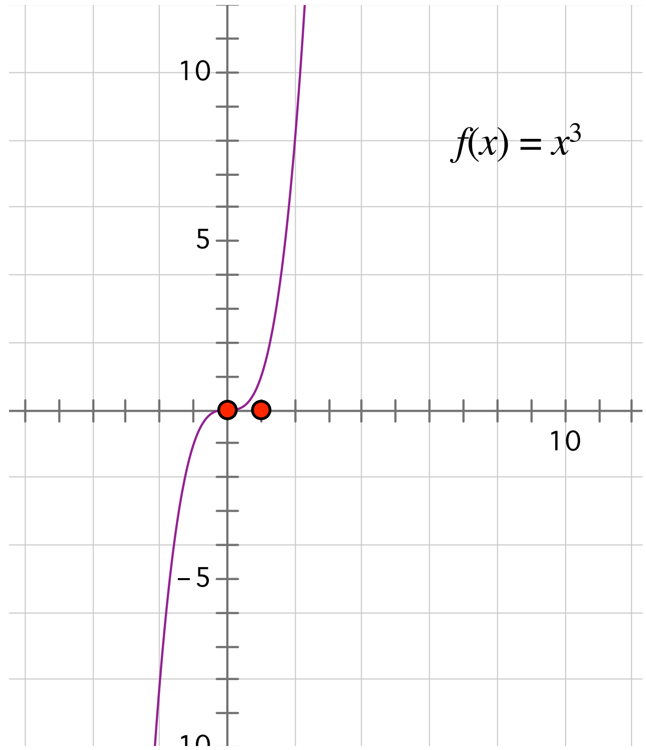
In reality, we may not have a formula for a function that occurs in nature. We may just have a table of values, showing the empirical data we have collected. Much of what we do in this course can be modified to get a substantial amount of information about the function from such a table of values.

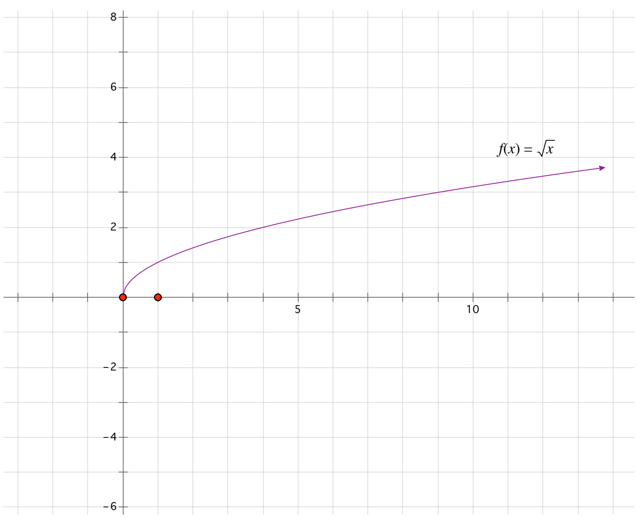
Sometimes we take the function that best fits a set of data (many software packages can help with this) and use that function to model the relationship between the variables and make predictions about the future.

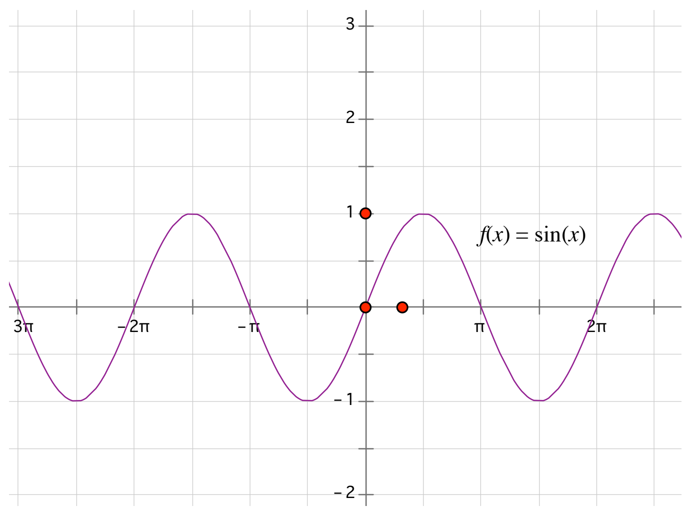
Appendix

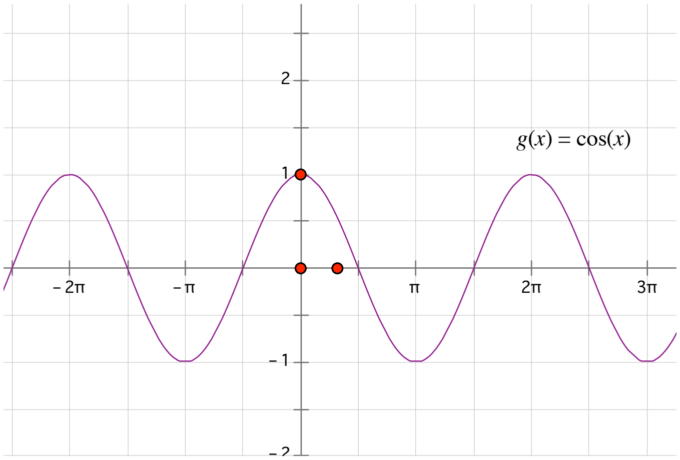
Catalogue of Graphs

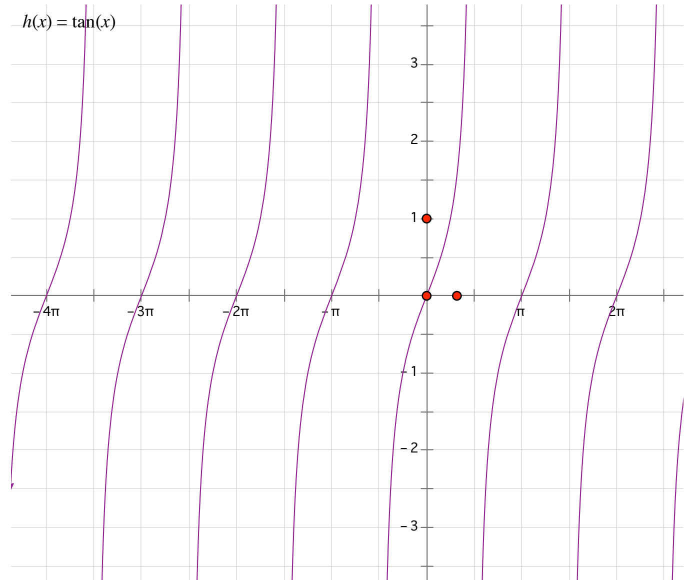


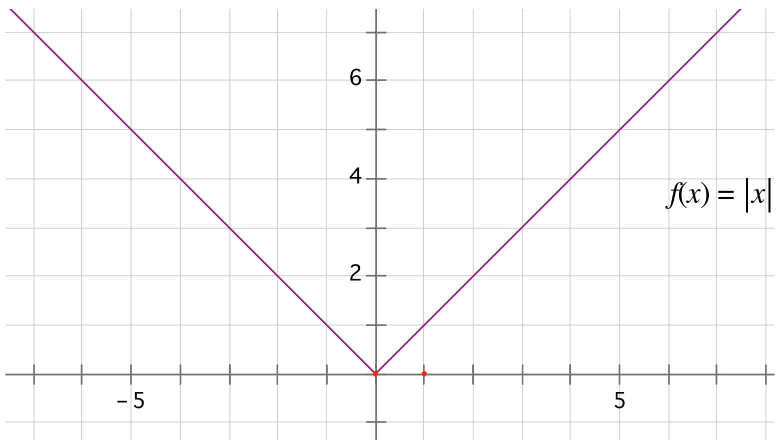






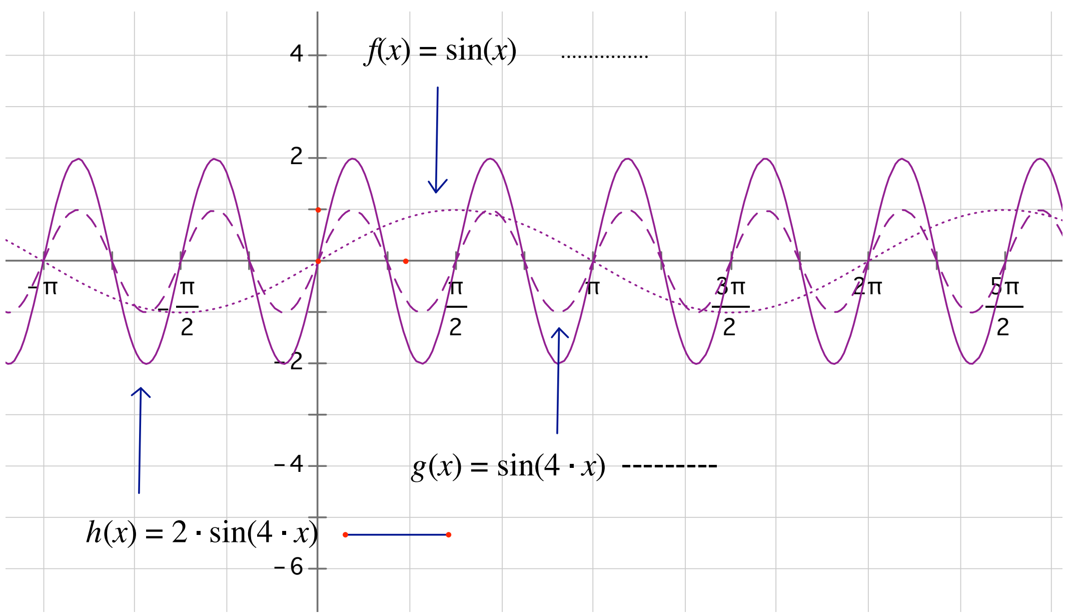






Further example of using graphing techniques

Example Sketch the graph of $y = 2\sin(4x)$. We see that this is a graph similar to that of $y = \sin(x)$, except with period $2\pi/4 = \pi/2$ and twice the amplitude:

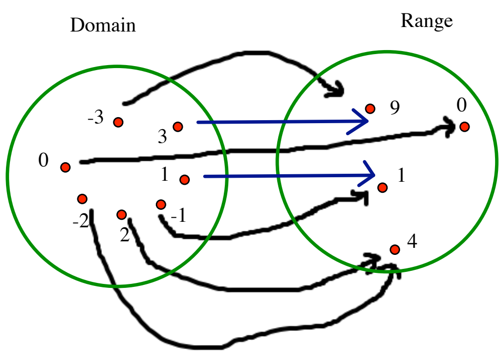


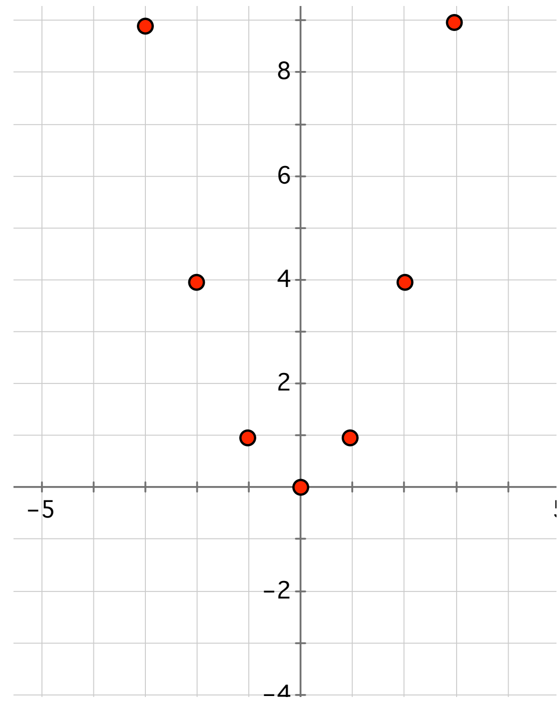
Visual representation by Venn Diagrams

This representation of function will not be used very much in this course, since our domain will usually contain an entire interval of the Real Number line and we cannot represent all of the points in the domain by an individual dot.

Example 3 Let f be the function which sends each of the numbers $-3, -2, -1, 0, 1, 2, 3$ to its square. We have $f(-3) = 9, f(-2) = 4, f(-1) = 1, f(0) = 0, f(1) = 1, f(2) = 4, f(3) = 9$.

We can represent this graphically in two ways. One is as in the set theoretic diagram below and the other is with a set of dots on the Cartesian plane, as in the diagram below.





Note that in the Venn Diagram, only one arrow leaves any given point in the domain. This reflects the fact that the function f assigns exactly one element to each element of the domain. There is no restriction on the number of arrows that may arrive at a point in the range.

Function Types

The most common types of functions, we shall deal with in this course are:

- **Polynomials** : functions of the form $f(x) = a_0 + a_1x + a_2x^2 + \dots a_nx^n$, where, $a_0, a_1, a_2, \dots, a_n$ are real numbers. Note that these functions include functions whose graphs are lines: $y = ax + b$, quadratics, $y = ax^2 + bx + c$, etc...

- **Rational Functions** These are functions of the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials, e.g $f(x) = \frac{x^2}{x-1}$.

- **Algebraic Functions** Anything constructed using $+, -, \cdot, \div, \sqrt{}$ starting with polynomial functions. e.g $g(x) = \frac{\sqrt{x^2+1}}{x^5+2x+5}$. or $h(x) = \sqrt{x + \sqrt{1+x}}$.

- **Trigonometric Functions** $\sin x$, $\cos x$, $\tan x$, $\sec x$, $\csc x$, $\cot x$. Note that the domains of these functions will always be in radians.